actual earth is close to a spherical stratification, so that Wavre's theorem, although theoretically applicable, is not "stable": a large change of the density law may go along with an unmeasurably small variation of the geometrical configuration.

Thus, of course, the density distribution of the earth can only be determined empirically: from seismology, free oscillations, etc.

3.2.4 Impossibility of a Purely Ellipsoidal Stratification

Consider the equation of an ellipsoid of revolution

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \quad . \tag{3–53}$$

Putting $A = 1/a^2$ and $B = 1/b^2$ we may write this as

$$A(x^2+y^2)+Bz^2-1=0 \quad . \tag{3–54}$$

To get a family of equisurfaces we must let A and B depend on a parameter, for which we may take the potential W:

$$A(W)(x^2+y^2)+B(W)z^2-1=0 \quad . \tag{3–55}$$

In fact, for any $W =$ const. we get some ellipsoid of the family.

An auxiliary formula. Eq. (3–55) has the form

$$F(x, y, z, W)=0 \quad . \tag{3–56}$$

If we express W as a function of the coordinates:

$$W=W(x, y, z) \tag{3–57}$$

and substitute into (3–56), we get an identity:

$$\bar{F}(x, y, z) \equiv F(x, y, z, W(x, y, z)) \equiv 0 \quad , \tag{3–58}$$

which may be differentiated (supposing smoothness) as often as we like. We differentiate twice ($F_x = \partial F/\partial x$, $F_W = \partial F/\partial W$, etc.)

$$\bar{F}_x = F_x + F_W W_x \equiv 0 \quad , \tag{3–59}$$

$$\bar{F}_{xx} = F_{xx} + 2F_{xW}W_x + F_{WW}W_x^2 + F_W W_{xx} \equiv 0 \quad . \tag{3–60}$$

Then we express W_x from (3–59):

$$W_x = -\frac{F_x}{F_W} \tag{3–61}$$

and substitute into (3–60), obtaining

$$F_{xx} - 2\frac{1}{F_W}F_xF_{xW} + \frac{1}{F_W^2}F_{WW}F_x^2 + F_W W_{xx} = 0 \quad . \tag{3–62}$$

Let us repeat our argument. Eq. (3–73) leads necessarily to (3–76) and thus excludes any ellipsoidal stratification that is not homothetic, i.e., that does not consist of geometrically similar ellipsoids. Then (3–83) shows that the density must be homogeneous, which excludes heterogeneous equilibrium figures with ellipsoidal stratification. This proves the

Theorem of Hamy–Pizzetti

> An ellipsoidal stratification is impossible for heterogeneous, rotationally symmetric figures of equilibrium.

This is an extremely important "no–go theorem". The history of the subject starts with Hamy in 1887 and continues with work by Volterra in 1903 and Véronnet in 1912. The present method of proof was given by Pizzetti (1913, pp. 190–193) and essentially also used by Wavre (1932, pp. 60–61). We have tried to streamline it and to make every step explicit.

Later (secs. 4.2.4 and 6.4) we shall see that the terrestrial level ellipsoid, even with an arbitrary non–ellipsoidal internal stratification, cannot be an exact equilibrium figure, although it is extremely close to such a figure (Ledersteger's theorem).

3.2.5 Another Derivation of Clairaut's Equation

Although *rigorously*, the spheroidal equisurfaces are not ellipsoids, they are so *in linear approximation* (in f). Thus Wavre has used his equation (3–40) for a very elegant derivation of Clairaut' s equation. We put $\Theta_1 = 0$ (Pole P), $\Theta_2 = 90°$ (Equator E), and write, noting $N(t, 0) = 1$,

$$\begin{aligned} g(t, 0) &= g_P(t) , & N(t, 90°) &= N_E(t) , \\ J(t, 0) &= J_P(t) , & J(t, 90°) &= J_E(t) . \end{aligned} \tag{3–84}$$

The equisurfaces are (approximately!) ellipsoids of semiaxes $a(t)$ and $b(t) = t$, so that

$$a(t) = \frac{t}{1-f} = t\,(1 + f(t)) + O(f^2) \quad . \tag{3–85}$$

We further have

$$N_E(t) = \frac{da}{dt} = 1 + f(t) + tf'(t) \quad , \tag{3–86}$$

always disregarding $O(f^2)$. The ellipsoidal formulas of sec. 1.4 give the mean curvatures to our linear approximation:

$$J_P = \frac{1}{t}\,(1-2f) \; , \quad J_E = \frac{1}{t} \; , \tag{3–87}$$

so that (3–40), with (3–39), readily becomes

$$\frac{4\pi G\rho - 2\omega^2}{g_P(t)} = \frac{-t^2 f'' + 6f}{2t^2 f' + 2tf}$$